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On conical swirling flows in an infinite fluid

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The steady axisymmetric flow generated in an unbounded incompressible viscous fluid, of density ρ and kinematic viscosity ν, by torque-producing singularities with constant line density c along the semi-infinite line $\theta = 0$ of a spherical polar coordinate system (r, θ, ϕ) that was investigated by Paull & Pillow (1985*b*), is reconsidered. The numerical solution constructed revealed the following features. (i) For values of c up to about 46.9 there is only one solution where the axial component of the meridional flow is directed from $\theta = 0$ to $\theta = \pi$. This solution can be continued to all values of c. (ii) For $c > 46.9$ the system of equations allows bifurcation and two more solutions with a single separatrix are possible. (iii) For $c = \infty$ one of the two branches of the separatrix asymptotes to $\theta = \frac{1}{2}\pi$ and the other to $\theta = \pi$. The asymptotic solution for large c constructed by Paull & Pillow (1985*b*), where the meridional flow consists of two colliding flows, relates to the bifurcation solution where the separatrix asymptotes to $\frac{1}{2}\pi$ as $c \to \infty$.

1. Introduction

In three interesting papers Pillow and Paull (Pillow & Paull 1985; Paull & Pillow 1985*a*, *b*) analysed and characterized in detail several causes for conical flows in an unbounded viscous incompressible fluid. Such flows are axisymmetric and, according to these authors, can be generated by four independent singularities along the symmetry axis ($\theta = 0, \theta = \pi$) of a spherical polar coordinate system (r, θ, ϕ). These are: (*a*) a semi-infinite line of sources (or sinks) with constant line density, stretching from the origin to infinity; (*b*) a point force at the origin directed along the symmetry axis of the system and generating a source of momentum; (*c*) two antisymmetric (about the origin) semi-infinite line forces (along the semi-infinite lines $\theta = 0, \theta = \pi$) with line density inversely proportional to the distance from the origin; (*d*) a semi-infinite line, say the line $\theta = 0$, of swirl-producing singularities pointing along the line with constant line density c. The flow fields generated by causes (*a*)–(*c*) are meridional and have no azimuthal component. Cause (*d*) generates an azimuthal flow field which induces a meridional flow that in the nonlinear regime interacts with and modifies the azimuthal flow. The meridional velocity field has a logarithmic singularity along the half-line $\theta = 0$ and can be associated with the flow generated by a (logarithmically infinite) point force at the origin O that discharges axial momentum in the direction $\theta = \pi$.

In the third of these papers Paull & Pillow (1985*b*, hereafter referred to as PP) studied in detail the flow fields associated with cause (*d*) assuming that the strength L (in PP notation) of the ring-circulation-producing singularity at the origin is zero. For the definition of ring circulation see the three papers by Pillow and Paull (1985). For small values of c they constructed an analytic solution and for large c an asymptotic solution of the problem. The meridional flow for the analytic solution represents a one-cell configuration and for the asymptotic solution a two-cell one. PP conjectured that at a value of c larger than 3 the θ-component of the velocity changes sign and the flow develops an internal viscous layer so that the asymptotic solution for large c is a continuation of the analytic solution. It appeared to us that the one-cell solution should exist for all c and the situation is more complicated than that described by PP. A numerical solution of the problem has confirmed our ideas and revealed that the asymptotic solution constructed by PP is one of two solutions relating to tangent bifurcation that start at $c \approx 46.9$ and exist for all higher values of c. The purpose of this paper is to present the appropriate solutions for various values of c.

2. General equations of the problem

The general equations of the problem have been derived by several authors. Here they are cast in the format used by PP in the three papers mentioned above and also by Serrin (1972) and their details will not be repeated. Below we briefly summarize the main features of the equations.

Let p, ρ, ν and $\boldsymbol{v}$ denote the pressure, density, kinematic viscosity and velocity of an incompressible viscous fluid. In a spherical polar coordinate system (r, θ, ϕ), the velocity is assumed to be axisymmetric and in the steady state is given by

$$\boldsymbol{v} = \frac{\nu}{r}\left[-g'(\mu), -\frac{g(\mu)}{(1-\mu^2)^{1/2}}, \frac{c\Omega(\mu)}{(1-\mu^2)^{1/2}}\right], \tag{1}$$

where $\mu = \cos\theta$, c is a positive constant and a prime denotes differentiation with respect to μ. The function $g(\mu)$ relates to a streamfunction ψ given by

$$\psi = \nu r g(\mu). \tag{2}$$

The pressure is given by

$$p = \nu^2 \rho h(\mu)/r^2. \tag{3}$$

If we substitute (1) and (3) in the momentum equation we find that its ϕ-component yields

$$(1-\mu^2)\,\Omega'' - g\Omega' = 0. \tag{4}$$

If we take the curl of the momentum equation, so as to eliminate the pressure, after some manipulation we obtain

$$gg''' + 3g'g'' - (1-\mu^2)\,g^{\mathrm{iv}} + 4\mu g''' = -2c^2\Omega\Omega'/(1-\mu^2). \tag{5}$$

When (5) is integrated three times it yields

$$g^2 - 2(1-\mu^2)\,g' - 4\mu g = 4c^2 G(\mu), \tag{6}$$

where

$$G = \tfrac{1}{8}(1-\mu)^2\int_{-1}^{\mu}\frac{\Omega^2(t)\,\mathrm{d}t}{(1-t)^2} + \tfrac{1}{8}(1+\mu)^2\int_{\mu}^{1}\frac{\Omega^2(t)\,\mathrm{d}t}{(1+t)^2} + A\mu^2 + B\mu + C, \tag{7}$$

A, B and C being constants of integration.

The boundary conditions are as follows:

(i) $\Omega(1) = 1$, $\Omega(-1) = 0$, due to the swirl singularities on $\mu = 1$ and the absence of them on $\mu = -1$.

(ii) Since there are no sources or sinks on the symmetry axis, $\psi = 0$ there, i.e. $g(1) = 0$, $g(-1) = 0$. These conditions yield

$$A + C = 0, \quad B = 0.$$

(iii) Since there are no singularities on the half-line $\mu = -1$, the velocity is finite there; in particular $g'(-1) = \text{finite}$. The conditions $g(-1) = 0, g'(-1) = \text{finite}$, mean that the left-hand side of (6) has a double zero at $\mu = -1$. Therefore $G(\mu)$ has a double zero at $\mu = -1$ and since $\Omega(-1) = B = 0$, equation (7) yields $A = C = 0$.

(iv) The torque-producing singularities generate ring circulation and the origin O is a source of axial momentum directed along $\mu = -1$ and of ring circulation. Here, as in PP, we assume that the strength L of the ring-circulation-producing singularity at O is zero. This condition determines the details of the singularity at O and is signified by the equation

$$c^2 = \int_{-1}^{1} \left[\mu \frac{g^2 + c^2(\Omega^2 - \frac{1}{4}(1+\mu)^2)}{1-\mu^2} - 2\mu g'(\mu)^2 - 6g(\mu)\right] d\mu. \tag{8}$$

Equation (8) is equation (4.14) on p. 386 of PP and details of its derivation can be found in that paper.

The function $h(\mu)$ occurring in the pressure term (see (3)) is obtained from the radial component of the momentum equation and it turns out to be

$$h = -\frac{1}{2}\left[\frac{g^2 + c^2\Omega^2}{1-\mu^2} + (gg' - (1-\mu^2)g'')'\right].$$

PP showed that this configuration is associated with a fictitious force along $\mu = 1$ pointing away from the origin with line density

$$-\pi c^2 \rho \nu^2 / r. \tag{9}$$

(In their notation the density of the fictitious force per unit length along $\mu = 1$ is $\rho K_A(1)/2r$; see their equation (7.1) on p. 374.) The same result is also obtained from the work of Goldshtik & Shtern (1990). They showed (on p. 486) that the force per unit length along $\mu = 1$ pointing in the direction of r increasing is

$$-4\pi\nu^2 \rho r^{-1} \lim_{\mu \to 1} [(1-\mu) g'']. \tag{10}$$

If we differentiate (6) and scrutinize the resulting expression we can see that (10) yields the expression given by (9).

The axial flow of the configuration can be associated with that of a point force F_0 at O pointing along $\mu = -1$. The point force can be evaluated as explained by Batchelor (1967, p. 209). After some manipulation we find that

$$\frac{F_0}{\pi \nu^2 \rho} = -\tfrac{1}{2}c^2 + \int_{-1}^{1} \left(\mu \frac{g^2 + c^2\Omega^2}{1-\mu^2} - 2\mu g'^2 - 6g\right) d\mu. \tag{11}$$

From (8) and (11) it can easily be shown that F_0 is logarithmically infinite. This is due to the fact that as

$$\mu \to 1, \quad p \sim -\frac{\nu^2 \rho c^2}{4r^2(1-\mu)} \quad \text{and in (11)} \quad \frac{\Omega^2}{1-\mu^2} \sim \frac{1}{2(1-\mu)}.$$

3. Solution of the problem

We set

$$g = -2(1-\mu^2)u'/u \tag{12}$$

and transform (6) into

$$u'' = c^2 G(\mu) u / (1-\mu^2)^2. \tag{13}$$

We substitute (12) in (4) and integrate twice subject to the boundary conditions $\Omega(1) = 1, \Omega(-1) = 0$, to obtain

$$\Omega = \int_{-1}^{\mu} u^{-2}(t)\,\mathrm{d}t \Big/ \int_{-1}^{1} u^{-2}(t)\,\mathrm{d}t. \tag{14}$$

The governing equations are (7) with $A = B = C = 0$, (13) and (14). These must be solved subject to (8) and in general must be solved iteratively. In view of (12), without loss of generality we can set $u(-1) = 1$. We specify a value of $u'(-1)$, say $u'(-1) = \alpha$ and vary α until the constructed solution satisfies (8).

Solution for small c

For small c a series solution in powers of c can be constructed by setting

$$g = c^2 g_1(\mu) + c^4 g_2(\mu) + \dots, \tag{15}$$

$$u = 1 + c^2 u_1(\mu) + c^4 u_2(\mu) + \dots, \tag{16}$$

$$\Omega = \Omega_0(\mu) + c^2 \Omega_1(\mu) + \dots. \tag{17}$$

From (16) and (14) we obtain $\Omega_0(\mu) = \frac{1}{2}(1+\mu)$ and then, from (7) and (13),

$$u_1'' = \frac{1}{8}\left[\frac{1}{1-\mu^2} + \frac{\ln[\frac{1}{2}(1-\mu)]}{(1+\mu)^2}\right]. \tag{18}$$

Integrating (18) and selecting the constant of integration so as to satisfy (8) to c^2, we obtain

$$u_1' = -\frac{1}{8}\left[\frac{1}{4} + \frac{\ln[\frac{1}{2}(1-\mu)]}{1+\mu}\right], \tag{19}$$

$$g_1 = \tfrac{1}{4}(1-\mu)\ln[\tfrac{1}{2}(1-\mu)] + \tfrac{1}{16}(1-\mu^2). \tag{20}$$

This solution was given by PP who also constructed the functions g_2 and Ω_1. These functions satisfy the conditions $g_1 < 0, \Omega_1 \geqslant 0$ and at some value of μ, say $\tilde{\mu}, g_2$ changes sign so that $g_2(\mu) > 0$ for $\mu > \tilde{\mu}$. PP suggested that the two-term expansion of (15) indicates that at some value of c, the function g associated with the one-cell configuration develops a zero in the range $-1 < \mu < 1$ and estimate (p. 377) that this occurs at a value a little larger than 3 and constructed an asymptotic solution for large c. The following argument indicates that the g-function associated with the one-cell solution does not change sign and the asymptotic solution may originate as a bifurcation at some specific value of c.

In view of (12), g will change sign at the value of μ where $u' = 0$. The function $G(\mu)$, given by (7), is positive except at $\mu = \pm 1$ where it is zero. Since $u > 0$, it follows from (13) that $u'' > 0$ and therefore u' increases monotonically as μ increases from -1 to $+1$. Therefore for u' to be zero (and g to change sign) it is necessary that $u'(-1) < 0$. It can be easily shown from (19) that $u_1'(-1) = 1/32$. PP showed that $g_2'(-1) \approx -0.014$, (their expression (5.21)), which in view of (12) and the condition $u(1) = 1$, yields $u_2'(-1) \approx 0.0035$.

Therefore for *small c*

$$u'(-1) = \frac{c^2}{32} + \frac{35c^4}{10000} + \dots,$$

that is the term associated with u_2 increases the value of $u'(-1)$ and indicates that u' increases as c increases and the function $g(\mu)$ does not change sign. The numerical solution presented below confirms this view.

The numerical solution

The numerical solution is constructed as follows. For a given c we guess $\Omega(\mu)$ and use (7) to construct $G(\mu)$; then we solve (13) subject to $u(-1) = 1, u'(-1) = \alpha$, where α is a fixed quantity, and construct $u(\mu)$ which we use in (14) for an improved $\Omega(\mu)$. The new $\Omega(\mu)$ is used in (7) for an improved $G(\mu)$ and so on until convergence. We assume that convergence has been achieved when at all points $|1-u_{n+1}/u_n| < 10^{-9}$, where $u_n(\mu)$ are the u-values obtained at the nth iteration. The solution constructed is then tested in (8). We varied the parameter α until (8) was satisfied. The integrals, occurring in (7), (8) and (14), were evaluated using the trapezoidal rule, and equation (13) was solved by Runge–Kutta methods. In general we used a step length of 0.0005.

The function g' occurring in (8) has a logarithmic singularity at $\mu = 1$ which can be integrated analytically. To evaluate (8) we express the quantities g and g' in terms of u, u' and u'' and for accurate results we found it necessary to split the range of integration into two subranges, $-1 \leqslant \mu \leqslant \mu_1$ and $\mu_1 \leqslant \mu \leqslant 1$, where μ_1 is close to 1. For the second subrange we set

$$1-\mu = x, \quad 1-\mu_1 = x_1, \quad u(\mu) = v(x).$$

In the second subrange the integral was evaluated partly analytically and partly numerically. The contribution of the second subrange to (8) was expressed as

$$\begin{aligned} &x_1(-32+52x_1)\,U_1^2+4x_1^2(3-x_1)\,U_1 \\ &+\int_0^{x_1}\left\{c^2(1-x)\frac{\Omega^2-(1-\frac{1}{2}x)^2}{x(2-x)}-8x^2(2-x)^2(1-x)\left(\frac{v''}{v}\right)^2\right. \\ &+x^2\left(\frac{v''}{v}-\left(\frac{v'}{v}\right)^2\right)\left[12-4x+(56-128x+32x^2)\frac{v'}{v}\right] \\ &\left.+x^2\left(\frac{v'}{v}\right)^2\left[-108+36x+(32-64x+40x^2-8x^3)\left(2\frac{v''}{v}-\left(\frac{v'}{v}\right)^2\right)\right]\right\}dx, \end{aligned} \tag{21}$$

where $U_1 = u'(\mu_1)/u(\mu_1)$ and the terms containing U_1 relate to the part of the integral done analytically.

It can easily be seen from (7) that near $\mu = 1$

$$G \approx \tfrac{1}{4}(1-\mu)[1+a(1-\mu)\ln(1-\mu)], \quad a = \Omega'(1).$$

Hence, for small x, (13) was approximated by

$$16xv'' = c^2(1+ax\ln x)\,v. \tag{22}$$

The two solutions of (22) (for small x) are

$$v_1 = x+c^2\left[\frac{1}{32}x^2+\frac{ax^3}{96}\ln x\right], \quad v_2 = 1+\frac{c^2}{16}\left[x+\left(\frac{a}{2}+\frac{c^2}{32}\right)x^2\right]\ln x.$$

To evaluate the integral in (21) we assumed that v'' is given by (22) and

$$\frac{v'}{v} = \frac{v_1'+Cv_2'}{v_1+Cv_2}, \tag{23}$$

where

$$C = \frac{v_1'(x_1)+U_1\,v_1(x_1)}{v_2'(x_1)+U_1\,v_2(x_1)} \neq 0, \tag{24}$$

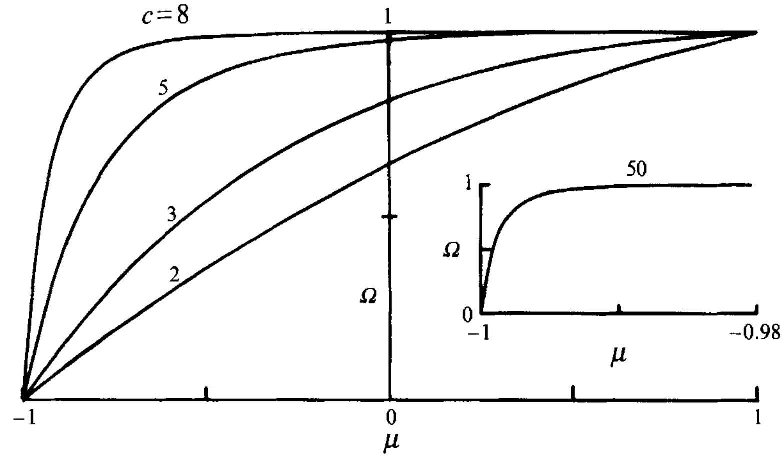

FIGURE 1. $\Omega(\mu)$ for some values of c, as shown on the curves.

that is for v we chose a linear combination of v_1 and v_2 so that at $x_1, v = u, v'(x) = -u'(\mu)$. In this procedure we set $x_1 = 0.001$.

As mentioned earlier the two-term expansion of g by PP indicates that as c increases so do the functions u, u' and u''. This indication was confirmed by our detailed computational results and it transpires that at a given μ the function $\Omega(\mu)$ increases as c increases. Indeed our computations showed that for a specified value of μ, say μ_2, at sufficiently large c, the value of Ω is practically 1 for $\mu > \mu_2$ and, as expected, μ_2 decreases when c increases. Values of Ω for some c are shown in figure 1.

For the case of large c where $\Omega = 1$ for $\mu > \mu_2$, an accurate evaluation of the integral occurring in (21) can be achieved. It can be shown from (7) that for this case

$$G = \tfrac{1}{4}[1-\mu-b(1-\mu)^2], \quad \mu > \mu_2, \tag{25}$$

where

$$b = \frac{1}{4}+\frac{1}{2(1-\mu_2)}-\frac{1}{2}\int_{-1}^{\mu_2}\frac{\Omega^2(t)}{(1-t)^2}\,\mathrm{d}t. \tag{26}$$

If we now set

$$1-\mu = x = \lambda y, \quad 1-\mu_1 = x_1 = \lambda y_1, \quad \lambda = 16/c^2, \quad u(\mu) = v(x) = w(y),$$

equation (13) for $\mu > \mu_2$ is transformed into

$$y(1-\tfrac{1}{2}y\lambda)^2 w'' = (1-by\lambda)\,w. \tag{27}$$

The two solutions of (27) are denoted by w_1 and w_2. For small y the first few terms of w_1 and w_2 were constructed analytically; thence w_1 and w_2 were integrated numerically (Runge–Kutta methods) to $y_1 = x_1/\lambda$. The function w is a linear combination of w_1 and w_2 so that at μ_1, $w = u$ and $w' = -\lambda u'$. The expression in the integral of (21) was converted into one involving y and w with w''/w given by (27). In the computations here we set $x_1 = 0.005$.

For large values of c the gradients of Ω and g are very large around $\mu = -1$. For this reason for $\mu < \mu_0$, where μ_0 is a constant close to -1, we used the transformation $1+\mu = \epsilon z$ and adjusted (7), (13) and (14), accordingly. For the data used we found, after some numerical experiments, that reasonably accurate results can be obtained by setting $\mu_0 = -0.98$, $\epsilon = 0.05$ and selecting a step length in the z-variable of 0.0005.

In figure 2 we have plotted the function $g/\text{Maximum}\,|g|$ for some values of c and, as

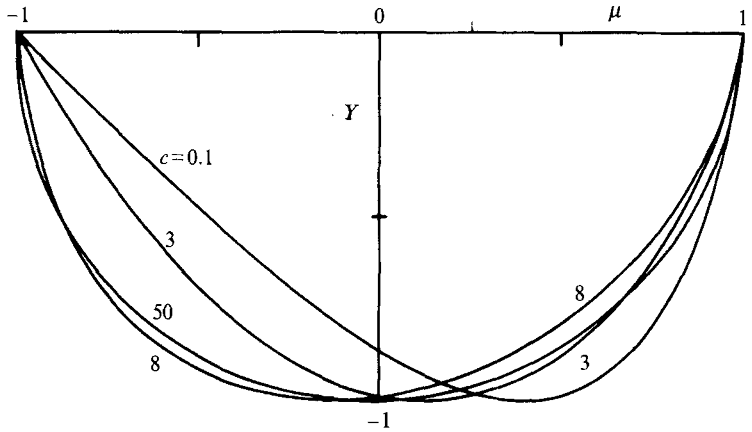

FIGURE 2. $g(\mu)/\text{Maximum}\,|g(\mu)| = Y$ for the values of c shown on the curves. For $c = 0.1, 3, 8, 50$ the maximum values of $|g(\mu)|$ are 0.00128, 1.48, 5.72, 35.39 and occur at $\mu = 0.40, 0.13, -0.13, -0.028$ respectively.

suggested earlier, this figure shows that g does not change sign. The case $c = 0.1$ represents, in effect, the one-term approximation $g \approx c^2 g_1$.

It can easily be shown, from (26), that the minimum value of b is 0.5 and occurs at $\mu_2 = -1$. We found that as c increases up to large values, $b \to 0.5$. For example for $c = 8$ and 50 the corresponding values of b are 0.5199 and 0.5003. Therefore for large c, (25) yields

$$G \approx \tfrac{1}{8}(1-\mu^2)$$

and, since then g is large, it follows from (6) that

$$g \approx -c(0.5(1-\mu^2))^{1/2}, \tag{28}$$

except near $\mu = \pm 1$, where there are boundary layers. The validity of the approximation (28) is confirmed by the function $g(\mu)$ shown in figure 2 for $c = 50$. For $c = 50$ the maximum value of $|g|$ obtained from the numerical solution is 35.39 at $\mu = -0.028$, whereas that obtained from (28) is 35.36 at $\mu = 0$.

Near $\mu = 1$, the asymptotic solution of (27), say $w_1 = 1 + y \ln y$. It then follows from (12) and the relationship between y and μ that the asymptotic value of g is given by

$$g \approx 4y \ln y. \tag{29}$$

More formally, near $\mu = 1$ the asymptotic solution of (27) for $\lambda \to 0$ is $y^{1/2} K_1(2y^{1/2})$, where K_n is the usual Bessel function. It then follows from (12) and the relationship between K_1 and K_0 that the asymptotic solution of g is given by

$$g = -4y^{1/2} K_0(2y^{1/2}) / K_1(2y^{1/2}). \tag{30}$$

This expression for g was given by PP in their equation (6.10) and for small y reduces to equation (29).

The quantity $g'(-1)$ is negative and for large c it is large. We found, for example, that for $c = 8$ and 20, $g'(-1) = -44$ and -435. The asymptotic forms of g and Ω near $\mu = -1$ for large c were constructed by Foster & Smith (1989) and by Goldshtik & Shtern (1990) and, in our notation, are

$$g = 4\eta/(4-\eta), \quad \Omega = -0.25g, \quad \eta = g'(-1)(1+\mu).$$

In figure 3 we have plotted the streamlines $\psi = $ constant in a meridian plane for

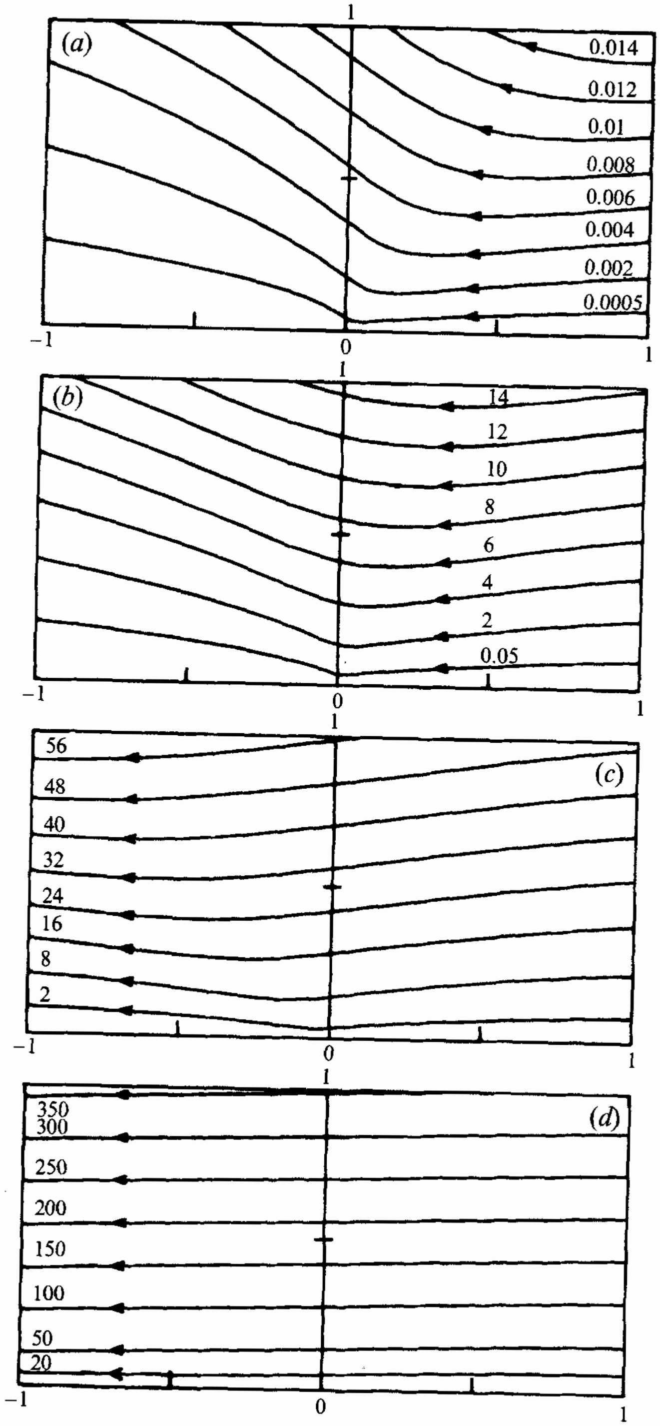

FIGURE 3. Meridian streamlines for some values of c: (*a*) $c = 0.1$, (*b*) $c = 3$, (*c*) $c = 8$, (*d*) $c = 50$. The torque-producing singularities are on the right half-axis of symmetry. The numbers on the curves are values of $-10\psi/\nu L$, where L is a characteristic length. The lengthscales are in the units of L.

some values of c. The case $c = 0.1$, in effect, relates to the Stokes flow regime where $g \approx c^2 g_1$. The streamline pattern for the case $c = 3$ is similar to that obtained by PP from the approximation $g \approx c^2 g_1 + c^4 g_2$. We find that for $c = 3$ near $\mu = -1$ the intensity of the flow is about 15% higher than that obtained from the two-term approximation of g. As c increases so does the overall intensity of the g-field, especially around the region $\mu = -1$, and at large c near the axis the g-field resembles a jet which exhibits convergence towards the line $\mu = -1$. (For $c = 50, g(0.999) \approx 0.5g(-0.999)$.)

The radial vorticity of the system, $\hat{\boldsymbol{r}} \cdot \nabla \times \boldsymbol{v}$, is $-\nu c \Omega'(\mu)/r^2$, that is the origin is a sink of vorticity with total vorticity input $2\pi\nu c\Omega(1) = 2\pi\nu c$. This vorticity is discharged in the form of a semi-infinite line vortex along $\mu = 1$ and induces an azimuthal velocity field such that the convective terms $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ in the momentum equation are rotational, that is the term $\Omega\Omega'$ in (5) is non-zero. The term $\Omega\Omega'$ in (5) induces a suitable g-field so that the total meridional flow balances it.

When $c \ll 1$, the radial vorticity is independent of μ and the meridional flow does not affect the vorticity distribution that generates it. As c increases (and the nonlinearities of the system become significant) interaction occurs between the meridional flow and the vorticity that generates it and, owing to convection, the radial vorticity density $\Omega'(\mu)$ is greatest near $\mu = -1$ and least near $\mu = 1$. At high values of c the radial vorticity of the system is concentrated in a conical region about the line $\mu = -1$ which becomes thinner as c increases. We found, for example, that for $c = 20$ there is no radial vorticity outside a cone (about $\mu = -1$) with semi-vertical angle 60° and for $c = 50$ there is no radial vorticity outside a cone with semi-vertical angle 25°. Since, by definition, $\Omega(-1) = 0$ the configuration described here will exist for all finite c. This conclusion is also reached from the work of Goldshtik & Shtern (1990, p. 489). Their reasoning is as follows. If we integrate (13) from $\mu = 1$ to $\mu = -1$ and there is a critical stage associated with the configuration it will manifest itself as $u(-1) = 0$ and $g(-1) = -4$ (see (12)). At that stage the outer solution of (13) will satisfy the conditions $u(1) = 1$, $u'(1) = 0$ and $u'(-1) = 0$. Since $G \geqslant 0$, if we satisfy the conditions $u(1) = 1$, $u'(1) = 0$ we cannot satisfy the condition $u(-1) = 0$ which implies that there is no critical configuration.

4. The bifurcation solutions

The asymptotic solution constructed by PP at large c represents two colliding flows along $\mu = \mu_* = -6 \ln c/c$ and relates to the case $\Omega = 0, g > 0$ for $\mu < \mu_*$ and $\Omega = 1$, $g < 0$ for $\mu > \mu_*$. That solution can only result from bifurcation. Indeed the integral immediately before their equation (6.12) on p. 372 yields

$$\mu_*(1-\mu_*^2)^{1/2} = -6 \ln c/c. \tag{31}$$

Therefore for a sufficiently large c there are two negative values of μ_* and as $c \rightarrow \infty$ one of these tends to zero (this one relates to the solution given by PP) and one tends to -1. It can easily be shown that the two solutions of (31) coalesce at $\mu_* = -1/\surd 2$ where the corresponding value of $c \approx 45.9$.

If we set $\Omega = 0$ for $\mu < \mu_*$ and $\Omega = 1$ for $\mu > \mu_*$, (13) becomes

$$u'' = \frac{c^2}{16} \frac{1-\mu_*}{1+\mu_*} \frac{u}{(1-\mu)^2}, \quad \mu < \mu_*, \tag{32a}$$

$$u'' = \frac{c^2}{4} \frac{1-b(1-\mu)}{(1-\mu)(1+\mu)^2} u, \quad \mu > \mu_*, \tag{32b}$$

where $b = \frac{1}{4} + \frac{1}{2}[1/(1-\mu_*)]$.

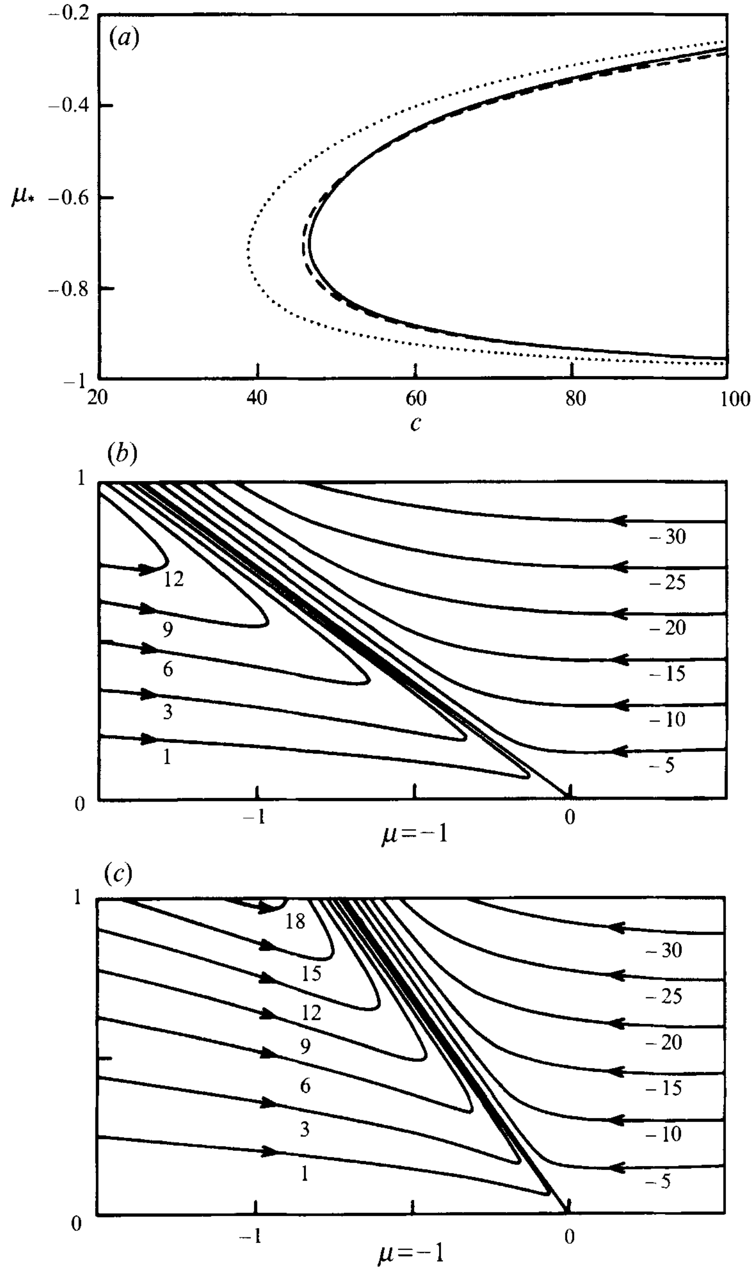

FIGURE 4. (*a*) Values of μ_* in terms of the parameter c: ----, asymptotic values given by (31); ..., values obtained from the solution of (32); ——, values obtained from the exact solution. (*b*) Colliding flows in a meridian plane (from the exact solution) relating to $c = 50$ and $\mu_* = -0.807$. (*c*) Colliding flows in a meridian plane (from the exact solution) relating to $c = 50$ and $\mu_* = -0.582$. The numbers on the curves in (*b*) and (*c*) are values of $\psi/\nu L$, where L is a characteristic length. The torque-producing singularities for (*b*) and (*c*) are on the right half-axis of symmetry.

Equations (32) were solved as follows. We specified c, guessed a value for μ_* and constructed a simple analytical solution of (32*a*) subject to $u(-1) = 1, u'(\mu_*) = 0$. Equation (32*b*) was then integrated numerically (from $\mu = \mu_*$ to $\mu = 1$) by the methods described in §3. The parameter μ_* was varied until the resulting solution satisfied (8). The minimum value of c, obtained by this method, is 39 and the corresponding value of $\mu_* \approx -0.724$. The function $\Omega(\mu)$, constructed from (14) and the solution of (32), is small for $\mu < \mu_*$ and as μ increases beyond μ_* it rapidly reaches the value of 1.

To construct a more accurate solution of the problem we must solve (7), (13) and (14). Since for large c it is difficult to solve equations such as (32*a*) by forward integration subject to $u(-1) = 0, u'(\mu_*) = 0$, we proceeded as follows. For the region $\mu < \mu_*$ we started at $\mu = \mu_*$ and integrated (6) to $\mu = -1$ subject to $g(\mu_*) = 0$. In practice we stopped close to $\mu = -1$, set $g(-1) = 0$ and extrapolated $g'(-1)$. The expression u^{-2} for $\mu < \mu_*$ (that is used in (14)) was obtained from (12), that is

$$u^{-2} = \exp \int_{-1}^{\mu} \frac{g(t)}{1-t^2} \mathrm{d}t.$$

In the region $\mu > \mu_*$ we made use of (13) and proceeded as described in §3. In this way for a given μ_* (and c) we constructed a convergent solution. We then varied μ_* until the constructed solution satisfied the boundary condition (8). The minimum value of c associated with this procedure is 46.9 and the corresponding value of $\mu_* = -0.704$. Figure 4(*a*) illustrates the values of μ_* in terms of c obtained from (31), the solution of (32) and the exact numerical solution of the problem. Figures 4(*b*) and 4(*c*) show the meridian streamlines for the two bifurcation solutions relating to $c = 50$. Both solutions represent colliding flows.

Since the point force at the origin points in the same direction ($\theta = \pi$) as the fictitious force along the half-line $\theta = 0$, the two bifurcation two-cell solutions that branch off at the origin and represent colliding flows must be due to the centrifugal effects of the flow and the discontinuity in the gradient of the torque-producing singularities at the foot of the semi-infinite line vortex.

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